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On the curvature and torsion of an isolated vortex filament

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We consider a very thin vortex filament in an unbounded, incompressible and inviscid fluid. The filament is not necessarily plane. Each portion of the filament moves with a velocity that can be approximated in terms of the local curvature of the filament. This approximation leads to a pair of intrinsic equations giving the curvature and the torsion of the filament, as functions of the time and the arc length along the filament. It is found that helicoidal vortex filaments are elementary solutions, and that they are unstable.

The intrinsic equations also suggest a linear mechanism that tends to produce concentrated torsion and a non-linear mechanism tending to disperse such singularities.

1. Introduction

This paper is motivated by a desire to understand simple three-dimensional flows endowed with vorticity. For simplicity, we assume that the fluid is incompressible and inviscid. For the same reason, we assume that the vorticity is concentrated along a single filament. If this single filament is straight or circular, we have a well-known situation, and therefore the next step in difficulty must introduce the concept of a variable curvature of the filament. As we shall see, a variable curvature produces torsion and the two phenomena are closely coupled.

Let us consider a vortex filament having a constant cross-section πa^2 (see figure 1) and a circulation Γ. The filament is transported by the velocity induced by its own vorticity. At any point in space, the induced velocity $\mathbf{V}$ can be calculated by the Biot–Savart formula

$$\mathbf{V} = \frac{1}{4\pi}\int \frac{\boldsymbol{\omega}\times\mathbf{r}}{r^3}\,d\xi^3, \tag{1.1}$$

where $\boldsymbol{\omega}$ is the vorticity and $\mathbf{r} = \mathbf{x}-\boldsymbol{\xi}$. If this equation is applied to the case of a circular vortex ring of radius R in the limit $a/R\rightarrow 0$, one finds that the velocity of the ring is constant and (Prandtl & Tietjens 1934) that it amounts to

$$V = \frac{\Gamma}{2\pi R}\ln\left(\frac{8R}{a\,e^{\frac{1}{4}}}\right). \tag{1.2}$$

It is remarkable that if $a\rightarrow 0$, the velocity becomes infinite. Physically, this is due to the fact that the net velocity induced at a point P (figure 1) is mainly the result of a particular type of contribution. Some contributions cancel each other, such as those of the two fluid elements A and A'. Others are weak because they come from distant fluid elements such as C and C'. Everything being considered, the leading contribution comes from fluid elements such as B and B' which are near the surface of a tangent cylinder (indicated by dashed lines in figure 1). If a vanishes while R and Γ stay constant, the influential elements such as B and B' move toward P and induce large velocities.

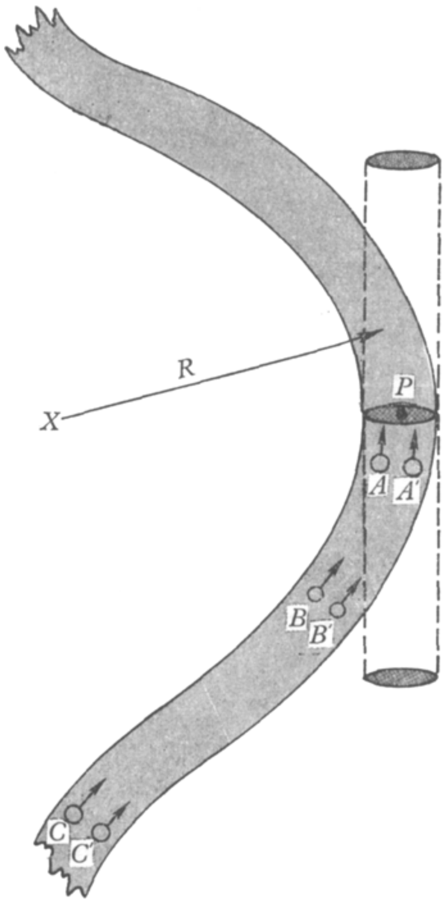

FIGURE 1. The vortex filament. The short arrows indicate the vorticity. At point P, the particles B and B' induce a velocity perpendicular to the page. Particles A and A' induce velocities of opposite directions which cancel out. Particles C and C' are too far away to be effective.

The same velocity is obtained at the point P if we simply assume that the filament is no longer circular, but that it is everywhere so thin that R/a is large. If s denotes the distance along the central line of the vortex filament, we can consider V and R as functions of s and of the time t. Since the logarithm varies slowly with the argument, a very good approximation to equation (1.2) is simply given by the relation

$$V = C/R, \tag{1.3}$$

where the velocity V is perpendicular to the tangent plane and where C is a constant. This equation has been previously obtained by Arms (1963, from private communication to Hama) for a plane filament, and used by Hama (1962, 1963). It can be shown that, in the limit of a small cross-section, it remains valid even if the filament is not plane. A suitable choice of the unit of time permits the reduction of C to unity. This paper is essentially concerned with the consequences of this approximation.

2. The intrinsic equations

If a vortex filament is plane but of variable curvature, the different portions of the filament will move normally to the plane at different velocities. Thus, the filament will acquire torsion. The numerical experiments of Hama (1962, 1963) illustrate this process very clearly. In general, if a curve moves under the influence of its own segments, two equations can be constructed specifying the evolution of the radius of curvature R and of the torsion T as functions of s and t: these are the intrinsic equations. They relate the two properties of the curve, without reference to any origin or orientation of the co-ordinates.

In order to obtain the intrinsic equations, we define $\mathbf{x}(s,t)$ as the vector locating the position of the filament in Cartesian co-ordinates. With a prime (′) indicating $\partial/\partial s$, and a dot (·) indicating $\partial/\partial t$, the tangent to the filament is $\mathbf{x}'$ and equation (1.3) becomes

$$\dot{\mathbf{x}} = \mathbf{x}' \times \mathbf{x}'', \tag{2.1}$$

where the cross-product assigns the correct directions to the velocity. Note that a time reversal is equivalent to a mirror image. By the definition of the tangent, we have

$$\mathbf{x}'.\mathbf{x}' = 1. \tag{2.2}$$

A differentiation gives

$$\mathbf{x}'.\mathbf{x}'' = 0, \tag{2.3}$$

which reminds us that the principal normal is orthogonal to the tangent. The radius of curvature R is given by

$$K = \mathbf{x}''.\mathbf{x}'' = 1/R^2. \tag{2.4}$$

We will use the following quantities:

$$\Delta = |\mathbf{x}', \mathbf{x}'', \mathbf{x}'''| = \mathbf{x}'.(\mathbf{x}'' \times \mathbf{x}'''), \tag{2.5}$$

$$T = \Delta/K, \tag{2.6}$$

$$M = \mathbf{x}'''.\mathbf{x}''', \tag{2.7}$$

where T is the inverse of the radius of torsion. Let us now derive some useful relations. Three successive differentiations of (2.3) lead to

$$\mathbf{x}'.\mathbf{x}''' = -K, \tag{2.8}$$

$$\mathbf{x}'.\mathbf{x}^{\text{iv}} = -\tfrac{3}{2}K', \tag{2.9}$$

$$\mathbf{x}'.\mathbf{x}^{\text{v}} = -K'' - M - 2\mathbf{x}''.\mathbf{x}^{\text{iv}}, \tag{2.10}$$

where roman numerals indicate fourth and fifth derivatives. Similarly, (2.7) gives

$$\mathbf{x}'''.\mathbf{x}^{\text{iv}} = \tfrac{1}{2}M'. \tag{2.11}$$

From (2.4) we obtain successively

$$\mathbf{x}''.\mathbf{x}''' = \tfrac{1}{2}K', \tag{2.12}$$

$$\mathbf{x}''.\mathbf{x}^{\text{iv}} = \tfrac{1}{2}K''-M, \tag{2.13}$$

$$\mathbf{x}''.\mathbf{x}^{\text{v}} = \tfrac{1}{2}K'''-\tfrac{3}{2}M'. \tag{2.14}$$

We can now consider time derivatives such as

$$\dot{K} = 2\dot{\mathbf{x}}''.\mathbf{x}'' = 2(\mathbf{x}'\times\mathbf{x}^{\text{iv}}).\mathbf{x}'' = -2\Delta', \tag{2.15}$$

and obtain the *first intrinsic equation* in the form

$$\dot{K} = -2(K'T+KT'). \tag{2.16}$$

Starting from (2.5) and (2.1), we now arrive at the relation

$$\begin{aligned}\dot{\Delta} = {} & 2(\mathbf{x}'.\mathbf{x}'')(\mathbf{x}''.\mathbf{x}^{\text{iv}})-2(\mathbf{x}''.\mathbf{x}'')(\mathbf{x}'.\mathbf{x}^{\text{iv}})\\ & +(\mathbf{x}'.\mathbf{x}')(\mathbf{x}''.\mathbf{x}^{\text{v}})-(\mathbf{x}'.\mathbf{x}'')(\mathbf{x}'.\mathbf{x}^{\text{v}})\\ & -(\mathbf{x}'.\mathbf{x}')(\mathbf{x}'''.\mathbf{x}^{\text{iv}})+(\mathbf{x}'.\mathbf{x}''')(\mathbf{x}'.\mathbf{x}^{\text{iv}}).\end{aligned} \tag{2.17}$$

Substitutions from (2.2), (2.3), (2.4), (2.8), (2.9), (2.11), and (2.14) lead to

$$\dot{\Delta} = \tfrac{1}{2}K'''+\tfrac{9}{2}KK'-2M'. \tag{2.18}$$

It now remains to establish a general relation between Δ and M. At any point of the filament, we can find three Cartesian axes such that $\mathbf{x}'$ lies along the first axis, and $\mathbf{x}''$ along the second. Then we denote by α, β, γ, the components of $\mathbf{x}'''$. By virtue of (2.3), (2.4), (2.5), and (2.7), we have

$$\mathbf{x}'.\mathbf{x}''' = \alpha, \tag{2.19}$$

$$\mathbf{x}''.\mathbf{x}''' = \beta K^{\frac{1}{2}}, \tag{2.20}$$

$$|\mathbf{x}',\mathbf{x}'',\mathbf{x}'''| = \Delta = \gamma K^{\frac{1}{2}}. \tag{2.21}$$

It follows from (2.7), (2.8) and (2.12) that

$$M = \alpha^2+\beta^2+\gamma^2 = K^2+\frac{1}{4}\frac{K'^2}{K}+\frac{\Delta^2}{K}. \tag{2.22}$$

This relation is valid in any system of co-ordinates. Let us now express Δ in terms of the torsion T (equation (2.6)). With some use of (2.16) and (2.22), we can transform (2.18) into the following *second intrinsic equation*

$$\dot{T}+2TT' = \frac{1}{2}\left(K'+\frac{K'''}{K}+\frac{K'^3}{K^3}-2\frac{K'K''}{K^2}\right). \tag{2.23}$$

3. Specific properties of the intrinsic equations

The intrinsic equations (2.16) and (2.23) constitute a badly non-linear system. However, it is obvious from (2.15) that, if Δ vanishes at some limits or if the filament is closed on itself, then

$$\frac{d}{dt}\int_{s_1}^{s_2} K\,ds = 0, \tag{3.1}$$

which indicates the conservation of a total curvature.

It is also apparent from (2.18) that, under some conditions,

$$\frac{d}{dt}\int_{s_1}^{s_2} \Delta\, ds = 0. \tag{3.2}$$

The meaning of this relation is mysterious, beyond the remark that a plane filament will develop equal and opposite amounts of the quantity Δ.

The equations also indicate that if K' and T' are everywhere zero at some time, they will remain zero at later times. This means that a helical vortex filament, having constant R and T will satisfy (2.1). Since R is constant, it will comply

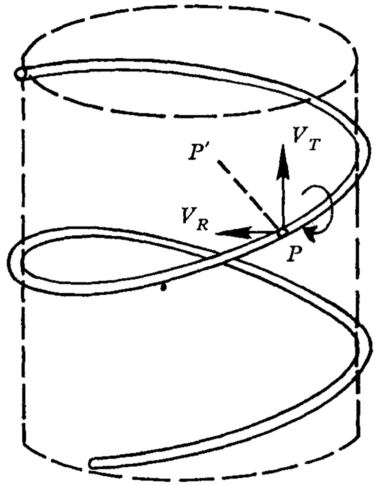

FIGURE 2. Motion of a helicoidal filament; point P moves toward P'.

even with (1.2), provided that a is constant. In space, the helicoidal filament will move with a translation velocity V_T and a rotation producing a tangential velocity V_R. The following relation can be found:

$$V_R/V_T = T/K^{\frac{1}{2}}. \tag{3.3}$$

The direction of motion is shown in figure 2. Incidentally, two intertwined, thin filaments of opposite vorticity will move toward each other.

If $\Delta = 0$, the filament is plane and the only solution is given by

$$\dot{K} = 0, \tag{3.4}$$

$$\frac{K'''}{K} + \frac{K'^3}{K^3} - 2\frac{K'K''}{K^2} + K' = 0. \tag{3.5}$$

Equation (3.5) can be integrated twice, with the result

$$s - s_0 = \pm\int_{s_0}^{s} \frac{dK}{(cK + bK^2 - K^3)^{\frac{1}{2}}}, \tag{3.6}$$

where b and c are integration constants. If $c \neq 0$, the ends of the filament cannot be straight. Indeed, the limit $K \to 0$ leads to $K = \frac{1}{4}c(s-s_0)^2$. If $c = 0$, the equation can be integrated exactly to

$$K = b\left[\cosh\frac{b^{\frac{1}{2}}}{2}(s-s_0)\right]^{-2}. \tag{3.7}$$

The constant b controls the overall size of the flow and the choice of $s_0 = 0$, $b = 4$ is convenient and immaterial. Then the shape of the filament is given by

$$x_1 = s - 2\tanh s, \tag{3.8}$$

$$x_2 = 2/\cosh s. \tag{3.9}$$

This curve is shown in figure 3. In the course of time, the filament simply rotates about the x-axis with a velocity proportional to $K^{\frac{1}{2}}$ which is proportional to x_2. Obviously, the crossing point requires some reappraisal of the whole approach. The solution is acceptable only if there is some slight torsion, otherwise we should include an interaction between two portions of the filament having two widely separated values of s. This was excluded from equation (2.1).

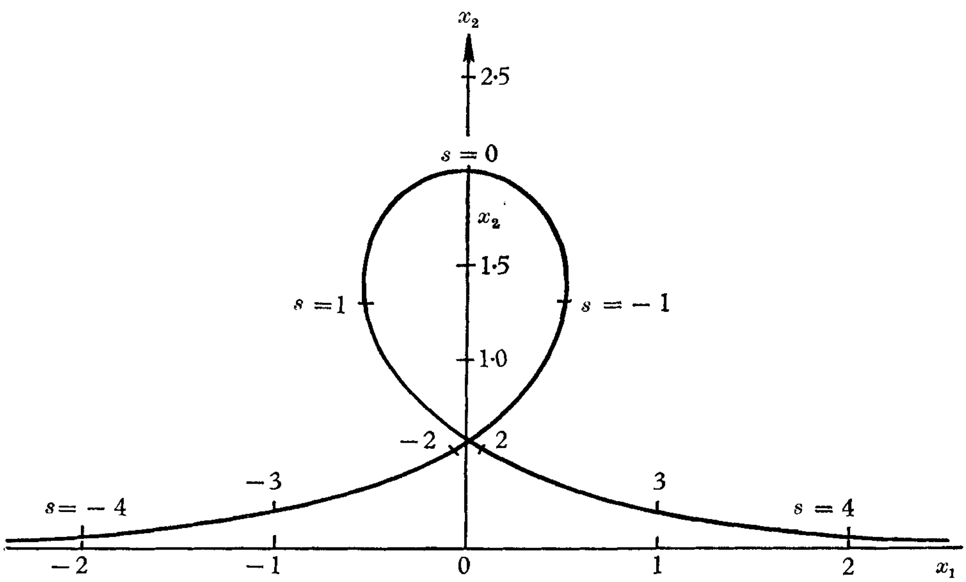

FIGURE 3. A stationary and plane vortex filament rotating about the x_1-axis.

Another property of vortex filaments is that, if the curvature is such that $K = (A\cos ks)^2$, the right-hand side of (2.23) reduces to the term $\frac{1}{2}K'$ for any value of A and k. If the filament is plane and has a sinusoidal shape with an amplitude much smaller than the wavelength, the above reduction occurs and, furthermore, the term $\frac{1}{2}K'$ can be neglected. In this approximation, the filament remains plane and rotates slowly, as found by Kelvin (1880).

Let us now consider the stability of a helicoidal filament. We assume that K and T are constant with small fluctuations κ and τ. By linearization of the basic equations, one obtains

$$\dot{\kappa} + 2T_0\kappa' = -2K_0\tau', \tag{3.10}$$

$$\dot{\tau} + 2T_0\tau' = \tfrac{1}{2}[\kappa' + (\kappa'''/K_0)]. \tag{3.11}$$

The terms in $2T_0\,\partial/\partial s$ on the left-hand side of (3.10) and (3.11) signify that the perturbations propagate along the filament with a velocity $2T_0$. (In ordinary units, this velocity is $(T_0\Gamma/\pi)\ln(8R/ae^{\frac{1}{4}})$, since T_0 has the dimension of (length)$^{-1}$

and Γ is the circulation.) We can either assume $T_0 = 0$ or shift to a new arc length $s - 2T_0 t$ in order to eliminate the terms in T_0. The results then lead to

$$\ddot{\kappa} + \kappa^{\text{iv}} + K_0 \kappa'' = 0, \tag{3.12}$$

$$\ddot{\tau} + \tau^{\text{iv}} + K_0 \tau'' = 0, \tag{3.13}$$

$$\dot{\kappa} = -2K_0 \tau'. \tag{3.14}$$

If the perturbation has a wavelength λ, and a pulsation Ω, (3.12) or (3.13) give

$$\Omega^2 = \left(\frac{2\pi}{\lambda}\right)^2 \left[\left(\frac{2\pi}{\lambda}\right)^2 - K\right]. \tag{3.15}$$

Thus, the perturbation propagates if $\lambda < 2\pi R$ and it grows exponentially if $\lambda > 2\pi R$. (Note that $2\pi R$ is greater than the circumference of the circle projected by the helix.) The most unstable wavelength is $\sqrt{2}(2\pi R)$. A circular vortex ring is neutrally stable, by the present argument.

4. General properties of the intrinsic equations

In this section, we shall discuss the properties of K and T in two extreme cases. In the first case, we shall assume that all the terms in (2.23) that contain derivatives higher than the first one are negligible. This means that there are only gradual variations of curvature and torsion. The equations can be written as follows:

$$\dot{K} + 2TK' + 2T'K = 0, \tag{4.1}$$

$$K(\dot{T} + 2TT') - \tfrac{1}{4}(K^2)' = 0. \tag{4.2}$$

If we regard K as the density ρ of some compressible fluid and $2T$ as its velocity V along the x-axis, (4.1) expresses the conservation of mass, and (4.2) indicates that the forces are produced by a pressure $-K^2/2$. Thus, this fictitious fluid moves toward the overdense regions. The characteristics are imaginary and the system of (4.1) and (4.2) has an elliptical character. Small perturbations obey a Laplace equation instead of the usual hyperbolic equation of acoustics, thus

$$\ddot{T} + T'' = 0. \tag{4.3}$$

In the (s, t)-space, the solutions are shaped either by boundary conditions or by singularities. We shall assume that the boundaries are at infinity or without influence. Then the solution vanishes unless (4.1) and (4.2) are not everywhere equal to zero. If K and T are given at $t = 0$, a singularity can therefore be encountered at some later value of t and at some particular value of s (figure 4). Clearly, such a singularity would involve derivatives higher than the first and would reintroduce the terms we neglected in the process of obtaining (4.1) and (4.2). This suggests that isolated singularities can emerge out of well-behaved initial conditions. This is to be expected from the analogy with a fictitious fluid of negative pressure.

Let us now consider the other extreme case when the highest derivatives become the leading terms. The second intrinsic equation then reduces to

$$\dot{T} = \tfrac{1}{2}K'''/K. \tag{4.4}$$

The equation for $\dot{K}$ (equation (2.16)) contains two terms of first order in the operator $\partial/\partial s$. The term in $T\,\partial/\partial s$ represents a kind of transport effect while the term in T' can lead to exponential growth. We shall therefore use the approximation

$$\dot{K} = -2KT'. \tag{4.5}$$

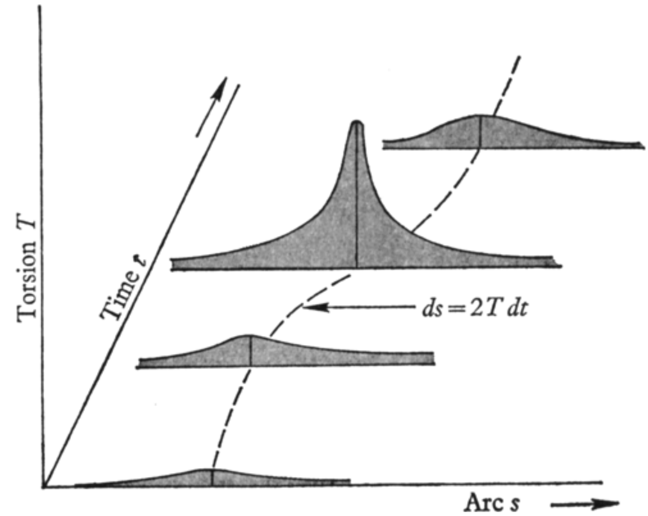

FIGURE 4. Production of a singularity in space-time, with elliptic equation.

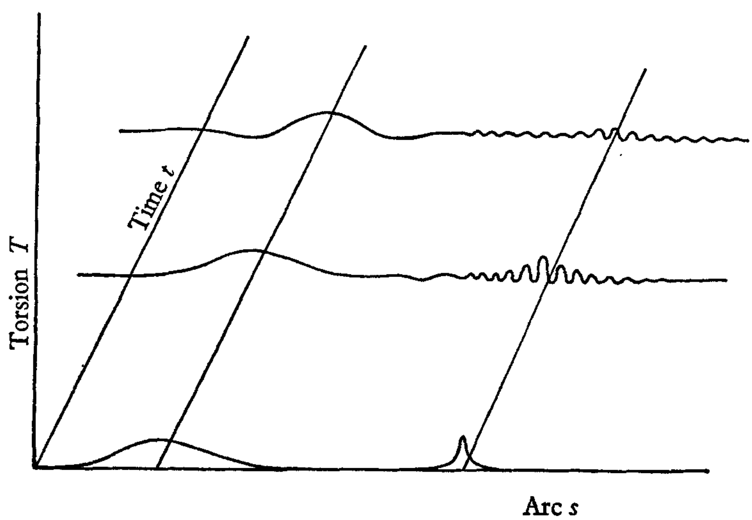

FIGURE 5. Dispersion of perturbations in space-time, according to (4.6).

If we now regard K as nearly constant, while K''' and T' are essential, we arrive at the equation

$$\ddot{T} + T^{\text{iv}} = 0. \tag{4.6}$$

This is the equation governing the lateral deformation of a thin rod. It has the basic property that perturbations of small wavelength are dispersed more rapidly than those of large wavelength, as illustrated in figure 5.

The existence of the two mechanisms described by (4.3) and (4.6) or figures 4 and 5 suggests that an isolated vortex filament starting from some random initial state may find a statistical equilibrium between production and dispersion of regions of concentrated torsion.

A mass of turbulent fluid can perhaps be considered as a system of entangled vortex filaments. If the interaction between filaments is weak in comparison with the intrinsic effects discussed in this section, the statistical properties of our intrinsic equations could be of interest. However, it is by no means certain that interaction between turbulent vortex lines can be neglected.

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